A Socialistic Approach to Construct Solution Operator for the Boltzmann Equation in a Small Bounded Domain

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The Boltzmann equation is an effect mathematical model for rarefied gas flows such as the gas flows with normal room conditions but confined in a very small domain. The phenomenon in such rarefied gas flows are very interesting and important in high technology industry; and they are very different from those in the inviscid compressible gas flows modeled by the compressible Euler equation or by the viscous flows modeled by the compressible Navier-Stokes equations. A quantitative and qualitative mathematical analysis on the Boltzmann equations would be an important key to understand the interesting phenomenon in the rarefied gas flows. The research described in this article is for the purpose to provide a rigorous qualitatively quantitatively mathematical tool for the Boltzmann equations with a small confined domain. This tool is the Green's function for the Boltzmann equations.

The Green's function is a generic and common tool to develop a rigorous quantitative and qualitative mathematical theory for the linear and nonlinear partial differential equations. This tool was invented by George Green (1793-1841), and it was first recognized by Lord Kelvin 1845 after the Green's death in 1840. Green's theorem and functions were important tools in classical mechanics, and were revised by Schwinger's 1948 work on electrodynamics that led to his 1965 Nobel prize. Green's functions is useful in analyzing superconductivity. On a visit to Nottingham in 1930, Albert Einstein commented that Green had been twenty years ahead of his time. The theoretical physicist, Julian Schwinger, who used Green's functions in his ground-breaking works, published a tribute, entitled "The Greening of Quantum Field Theory: George and I," in 1993.

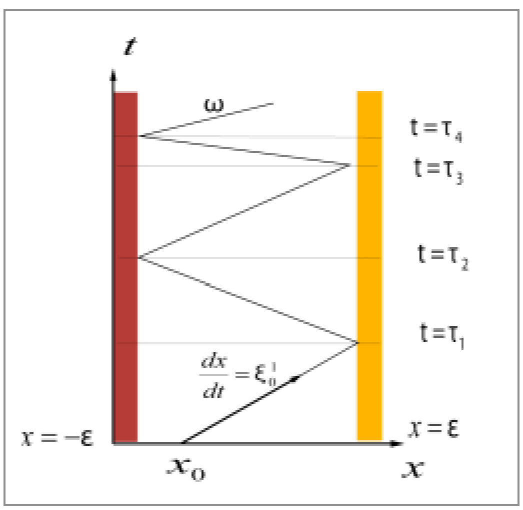

FIGURE 1

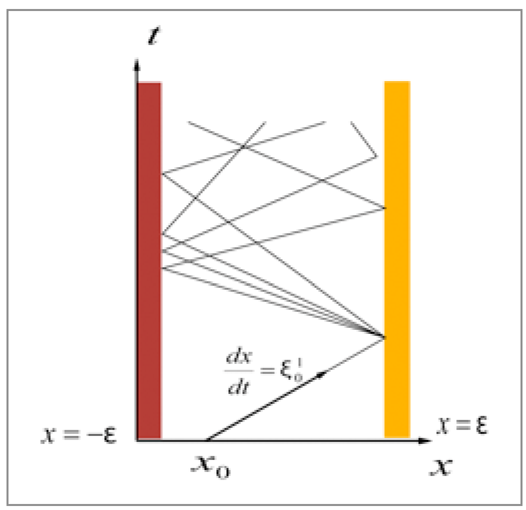

FIGURE 2

In mathematics, the Green's function had been developed for many differential types of partial differential equations such as wave equations, heat equations, etc. The methodology in constructing the Green's function through the spectral analysis became matured for many types of equations. However, this is not the case for the Boltzmann equation. One simple reason is that the Boltzmann equation is an infinite dimensional mesoscopic equation. This mesoscopic equation possesses dual structures, which are the microscopic flow structure (particlelike structure) and the macroscopic structure (dissipative wavelike structure), see [1]. This mesocopic phenomena prevents the spectral analysis approach to obtain the full feature of the Green's function. A particlelike-wavelike decomposition was introduced in [1] to develop the Green's function for the Boltzmann equation with a precise pointwise structure. This gives a precise qualitative and quantitative description on the transition of the mesoscopic flows from the particlelike state to the dissipative wavelike state for the Boltzmann equation. However, this decomposition does not apply to any problem with a diffuse boundary condition, which is used to model the roughness of the boundary, since the mechanism of the boundary did not present in the construction of the Green's function. A stochastic approach is introduced to construct the Green's function with the effect of the boundary counted. All mechanisms in the Boltzmann equation, the free transport mechanism, the particle-particle collision, and particle-boundary collision, are counted in the construction, see [2].

To apply a stochastic approach one will require the following two conditions:
(1) The ratio of the mean free path to the diameter of the confined domain is large enough.
(2) The boundary is rough enough so that the boundary is diffuse.

The first property is a mechanism to separate the particle-particle collision and particle-boundary collision. Under (1), most of the particles will collision boundary sufficient many times before they collide with another particle. Thus, one can introduce a suitable time scale so that during this time scale one considers the effect from the boundary and there are sufficient many collisions with the boundary within the given time scale. Under (2), the particle velocity distributions before collision and post collision are independent, (Figure 2). This independent property gives a Markovian processes, (Figure 1), for the system without particle-particle collision. Since in the given time scale there are sufficiently many collision taking place, a strong version central limit theory in probability can be applied to compute the accumulated effect due to the particle-boundary collision. The central limit theory is sufficient to compute the fluctuation of the collision frequency at the boundary, and it yields the convergent rate to the equilibrium state within the given time scale. Then, one can apply the conservation law, that is, total number of particles is conservative, to obtain the equilibrium state which the flow converges to. This results in a precise estimate of the solution of the free transport equation so that one will be able to compute the effect of the particle-particle collision within the given time scale. The particle-particle collision is counted as a large time scale dissipation mechanism. It regulates the flows towards its equilibrium state with an exponential rate. Finally, one obtains the Green's function with precise pointwise structure for all time scales. With the structure of the Green's function, one can obtain the solution of the fully nonlinear problem.

This stochastic approach points out the significance of the physical boundary for gas flows with very large Knudsen number. It also provides an effective mathematical tool to compute it. Such flows with large Knudsen number are common in a nano device, and the analysis in [2] has the potential to be converted into a computational tool for analyzing the industrial problems.

Publications:

[1] Lu, Tai-Ping;Yu, Shih-Hsien, **The Green's function and large-time behavior of soloutions for the one-dimensional Boltzmann equation**, Commun. Pure Appl. Math. Vol 57 (2004) no 12 1543-1608

[2] Yu, Shih-Hsien, **Stochastic formulation for the initial-boundary value problems of the Boltzmann equation.** Arch. Ration. Mech. Anal. 192 (2009), no. 2, 217--274.