INSTRUCTIONS TO CANDIDATES

1. Write your name here: ________________________________

2. This paper contains a total of SIX (6) questions and comprises TWENTY (20) printed pages, including this page.

3. This is a CLOSED BOOK test. No list of formulas is provided and helpsheets are disallowed.

4. Only non-programmable and non-graphing calculators without remote communication function may be used. However, you should lay out systematically the various steps in your calculations.

5. Candidates must answer ALL 6 questions.

6. Write your solutions in the spaces provided below the questions in this test paper. Submit this test paper at the end of the test period.
Question 1 [10 marks]

Find the values of $a$ and $b$, if any, such that the system

$$
\begin{cases}
ax + bz = 2 \\
ax + ay + 4z = 4 \\
ay + 2z = b
\end{cases}
$$

has

(a) no solution;

(b) exactly one solution;

(c) infinitely many solutions and the general solution has one arbitrary parameter;

(d) infinitely many solutions and the general solution has two arbitrary parameters.

Solution:
(More space for solution to Question 1.)
(More space for solution to Question 1.)
Question 2 [15 marks]

(a) Let $A = [a_{ij}]$ and $B = [b_{st}]$ be $n \times n$ upper triangular matrices, where $n \geq 2$. Show that $AB$ is also an $n \times n$ upper triangular matrix.

(b) Find all nonzero $2 \times 2$ matrices \[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\] such that \[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}^2 = 0.
\]

Solution:
(More space for solution to Question 2.)
(More space for solution to Question 2.)
Question 3  [20 marks]

(a) Find necessary and sufficient conditions on the constants $a$, $b$, $c$ and $d$ such that the matrix

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
a & b & c & d \\
a^2 & b^2 & c^2 & d^2 \\
a^3 & b^3 & c^3 & d^3
\end{bmatrix}
\]

is invertible. Justify your answer.

(b) Let $p$ be a constant and $Q = [q_{ij}]$ be an $n \times n$ matrix, with $n \geq 2$, such that

\[
q_{ij} = \begin{cases}
p & \text{if } i = j \\
1 & \text{if } i \neq j
\end{cases}
\]

Find $\det(Q)$ in terms of $n$ and $p$.

Solution:
(More space for solution to Question 3.)
(More space for solution to Question 3.)
(More space for solution to Question 3.)
Question 4 [15 marks]

Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a linear operator such that \( T^3(v) = 0 \) for all \( v \in \mathbb{R}^2 \). Determine whether \( T^2(v) = 0 \) for all \( v \in \mathbb{R}^2 \). Justify your answer.

Solution:
(More space for solution to Question 4.)
Question 5  [20 marks]
Let A and B be $m \times p$ and $p \times n$ matrices respectively.

(a) Show that the nullspace of $B$ is a subset of the nullspace of $AB$. Hence show that $\text{rank}(AB) \leq \text{rank}(B)$.

(b) Show that every column of the matrix $AB$ lies in the column space of $A$. Hence, or otherwise, show that $\text{rank}(AB) \leq \text{rank}(A)$.

(c) Suppose the linear system $Ax = b$ is consistent for any $b \in \mathbb{R}^m$. Show that the linear system $A^T y = 0$ has only the trivial solution.

(Notation: $A^T$ is the transpose of $A$.)

Solution:
(More space for solution to Question 5.)
(More space for solution to Question 5.)
Question 6 [20 marks]

(a) Let $p$ and $q$ be constants. Suppose the matrix

$$A = \begin{bmatrix} 1 & 2 & p \\ 0 & 3 & 1 \\ q & 5 & -1 \end{bmatrix}$$

has three distinct eigenvalues $\lambda_1$, $\lambda_2$ and $\lambda_3$. Find the sum

$$\lambda_1 + \lambda_2 + \lambda_3.$$

Justify your answer.

(b) Let $B$ be a $2 \times 2$ matrix with real entries such that $B^2 = B$. Determine whether $B$ is diagonalizable. Justify your answer.

Solution:
(More space for solution to Question 6.)
(More space for solution to Question 6.)
(More space for solution to Question 6.)

End of Test