NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

ADVANCED PLACEMENT TEST
(SAMPLE)

MA1101R LINEAR ALGEBRA

MMM-YYYY — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write your name here:** ____
2. This paper contains a total of **SIX (6)** questions and comprises **TWENTY (20)** printed pages, including this page.
3. This is a **CLOSED BOOK** test. No list of formulas is provided and helpsheets are disallowed.
4. Only non-programmable and non-graphing calculators without remote communication function may be used. However, you should lay out systematically the various steps in your calculations.
5. Candidates must answer **ALL** 6 questions.
6. **Write your solutions in the spaces provided below the questions in this test paper. Submit this test paper at the end of the test period.**

Question 1 [10 marks]
Find the values of a and b, if any, such that that the system

$$\begin{cases} ax \qquad + bz = 2 \\ ax + ay + 4z = 4 \\ \qquad ay + 2z = b \end{cases}$$

has

(a) no solution;

(b) exactly one solution;

(c) infinitely many solutions and the general solution has one arbitrary parameter;

(d) infinitely many solutions and the general solution has two arbitrary parameters.

Solution:

(*More space for solution to Question 1.*)

(*More space for solution to Question 1.*)

Question 2 [15 marks]

(a) Let $A = [a_{ij}]$ and $B = [b_{st}]$ be $n \times n$ upper triangular matrices, where $n \geq 2$. Show that AB is also an $n \times n$ upper triangular matrix.

(b) Find all nonzero 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = 0$.

Solution:

(*More space for solution to Question 2.*)

(*More space for solution to Question 2.*)

Question 3 [20 marks]

(a) Find necessary and sufficient conditions on the constants a, b, c and d such that the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{bmatrix}$$

is invertible. Justify your answer.

(b) Let p be a constant and $Q = [q_{ij}]$ be an $n \times n$ matrix, with $n \geq 2$, such that

$$q_{ij} = \begin{cases} p & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

Find $\det(Q)$ in terms of n and p.

Solution:

(*More space for solution to Question 3.*)

(*More space for solution to Question 3.*)

(*More space for solution to Question 3.*)

Question 4 [15 marks]
Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear operator such that $T^3(v) = 0$ for all $v \in \mathbb{R}^2$. Determine whether $T^2(v) = 0$ for all $v \in \mathbb{R}^2$. Justify your answer.

Solution:

(*More space for solution to Question 4.*)

Question 5 [20 marks]
Let A and B be $m \times p$ and $p \times n$ matrices respectively.

(a) Show that the nullspace of B is a subset of the nullspace of AB. Hence show that $\text{rank}(AB) \leq \text{rank}(B)$.

(b) Show that every column of the matrix AB lies in the column space of A. Hence, or otherwise, show that $\text{rank}(AB) \leq \text{rank}(A)$.

(c) Suppose the linear system $Ax = b$ is consistent for any $b \in \mathbb{R}^m$. Show that the linear system $A^T y = 0$ has only the trivial solution.

(*Notation:* A^T *is the transpose of* A.)

Solution:

(*More space for solution to Question 5.*)

(*More space for solution to Question 5.*)

Question 6 [20 marks]

(a) Let p and q be constants. Suppose the matrix

$$A = \begin{bmatrix} 1 & 2 & p \\ 0 & 3 & 1 \\ q & 5 & -1 \end{bmatrix}$$

has three distinct eigenvalues λ_1, λ_2 and λ_3. Find the sum

$$\lambda_1 \ + \ \lambda_2 \ + \ \lambda_3.$$

Justify your answer.

(b) Let B be a 2×2 matrix with real entries such that $B^2 = B$. Determine whether B is diagonalizable. Justify your answer.

Solution:

(*More space for solution to Question 6.*)

(*More space for solution to Question 6.*)

(*More space for solution to Question 6.*)

End of Test